

FOREST CARBON SEQUESTRATION AND OPTIMAL HARVESTING DECISION CONSIDERING SOUTHERN PINE BEETLE (SPB) DISTURBANCE: A REAL OPTION APPROACH

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Keywords

real option, flexible harvest, carbon sequestration, southern pine beetle, value of uncertainty

Abstract

This study evaluates the forest management decision making of loblolly pine forest in the southern U.S. using the real option approach. The study incorporates three uncertainties that forest owners have faced including timber price volatility, forest carbon sequestration, and impacts of insect outbreaks into the real option model to investigate the relationship between such uncertainties and forest bare land value and tree rotation age. The results show that forest owners can face a mixed outcome of these uncertainties when they make forest management decisions, and the real option approach helps the forest managers consider future consequence through allowing the flexible harvest decision. Generally, a higher bare land value is generated if a flexible harvest decision making (real option) is allowed compared to a fixed harvest. The standing tree sequesters CO₂, and the forest's role of carbon sequestration could generate extra value in the forest while insect outbreaks reduce the bare land value. The increasing social cost of carbon tends to call for increasing the bare land value of forest tree rotation age. Therefore, as climate change becomes more looming due to CO₂ concentration in the atmosphere, the value of standing forests would increase due to enhancing opportunity cost of carbon sequestration in forests. Continuous efforts of pest management for forests are necessary since a higher insect risk tends to reduce the bare land value of forests. Also, employing marketable climate policy such as emissions trading is necessary to create a market carbon price and offset extra cost to keep the forest.

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I. Introduction

Forest owners in the southern U.S. region are facing several risks, and these risks are increasing in magnitude with climate change. Uncertainties associated with management decisions are challengeable tasks of forest managers because inappropriate decision making can result in the loss of economic opportunities and profits due to the irreversible characteristic of forests. Moreover, the ongoing climate change tends to accelerate the uncertainties by altering forest disturbance and forest ecology. The fundamental challenges for forest resource management decision making are evaluating trade-offs between the social-economic benefit of harvesting timber and the ecological benefit of preserving the forests (Morgan, Abdallah, and Lasserre 2007). To examine this need, this paper investigates a developed methodology to adopt for forest management strategy under uncertainties. This study applies the real options valuation approach to the field of forest management decision making considering various cases that forest owners might face. The real option approach can supplement the main weakness of traditional forest management evaluation, because it takes into account the flexibility of harvest decisions due to timber price fluctuations (Tee et al. 2014). Also, the real option approach does fully consider the possibility of reversible investment opportunities (Duku-Kaakyire and Nanang 2004).

The definition of real option is the value of being able to choose some characteristic of decision allowing flexible outcome (Saphores 2000). The term "real" refers to tangible assets such as facilities and natural resource, and several studies have adapted a real options framework to the field of forestry. Developments in real option study in forestry have increased the need for risk management of forest investment and forest business management for optimizing the financial performance of forest assets.

The real option approach is very useful in understanding tradeoffs between timber and ecosystem services provided by forests to incorporate uncertainties and flexibility in timing (Alavalapati and Kant 2014). Tee et al. (2014) applied real options analysis of forestry carbon valuation under the New Zealand emission trading scheme. They incorporated both stochastic timber price and carbon value into calculating real option value of the New Zealand forests using the binomial tree method.

This study demonstrates the utility of the real options valuation approach to the field of forest management decision making considering various cases that forest owners might face. The term "valuation" means the bare land value of loblolly (*Pinus taeda*) pine plantation in southern U.S. Loblolly pine is the most commercially important forest species in the southern U.S., and its native range extends throughout 14 states from southern New Jersey to central Florida and to eastern Texas (Baker and Langdon 1990). The objectives of the study are to find answers to the following questions:

- 1) How does the sawtimber price volatility affect the bare land valuation of loblolly pine forests in the southern U.S.?
- 2) How much could the bare land value be changed if we consider not only timber price but also the carbon sequestration ability of the forest and pine beetle outbreak risk?
- 3) What is the optimal harvesting decision for loblolly pine plantations in the southern U.S. considering timber price volatility, carbon value, and pine beetle infestation risk?

This study applies binomial tree methods based on Guthrie's approach (2009), for evaluating real option value. The binomial tree methods have several advantages such as numerically efficient and conceptually undemanding technique to calculate option value. The main contribution of this study is to evaluate the optimal stand management decision considering timber price, carbon sequestration, and trees damaged by insects, southern pine beetle (SPB) in particular, which is one of the main causes of tree damages in the southern U.S. There are many studies that evaluate the value of the forests using the real options theory but researchers have not treated damaged trees in detail. Insect infestation directly affects forest owner's profit because it reduces timber productivity. Regarding forest carbon sequestration, dead trees do not release significant amounts of CO₂ into the atmosphere than expected because dead trees hold their carbon for a long time and prevent it from quickly being released into the atmosphere (Moore et al. 2013). Thus, damaged trees represent a substantial proportion of the total carbon sink/source in forest stands, and these damaged trees will affect tree management decision such as harvesting age (Asante, Armstrong, and Adamowicz 2011). Without considering this, the carbon sequestration ability of forest could be underestimated. This paper provides guidelines for forest owners for improving their timber harvest decisions to consider some cases they could face under cli-

mate change including timber price volatility risk, benefit from mitigation CO₂ due to forest carbon sequestration, and SPB outbreak risk.

2. Model setting up

1. Binomial tree of price movement

Timber price volatility is one of the critical uncertainties that forestland owners could face. Suppose that $X = (0,0)$ is the current price of sawtimber (\$/m³). $X(i,n)$ denotes the sawtimber price at the node (i, n) , where i is the number of downward price moves and n is the time step. Suppose that U, D are the size of the up movement and down movement where $U = e^{\hat{\sigma}\sqrt{\Delta tm}}$ and $D = e^{-\hat{\sigma}\sqrt{\Delta tm}}$ (see equation (A1) in appendix), respectively. Sawtimber price could be either increased or decreased with probability $\theta_U(i,n)$ or $\theta_D(i,n)$ at each node. If sawtimber price is increased at the node $(i,n+1)$, it could be $X(i,n+1) = X(i,n)U$ and $X(i+1,n+1) = X(i,n)D$ when sawtimber price is decreased. The binomial tree of sawtimber price movement process for $n=2$ is described in Figure A1. The forestland owners expect some profits from the sales of forest products; the amount of the profit depends on the timber price movement in the market. Assume that this timber price follows a mean-reverting series. Schwartz (1997) suggested a strong mean reversion in the commercial commodity prices. The mean-reverting price process implies that unlike the random walk price process, shocks to mean-reverting timber spot prices are not permanent. In other words, the sudden increase in timber price leads to an increase in supply as well, so the market price of timber will move back towards the timber's long-run marginal cost of production in long-term. Likewise, a sudden decrease in timber price causes a reduction in supply that triggers increase in future timber price. Therefore, a sudden increase (decrease) in timber spot price is not sustainable (Guthrie 2009).

Under the mean-reverting price assumption, the logarithm of the price follows a first order autoregressive process:

$$(1) \quad \begin{aligned} P_{j+1} - P_j &= \alpha_0 + \alpha_1 P_j + u_{j+1} \\ u_{j+1} &\sim N(0, \phi^2) \end{aligned}$$

where p_j is the market price of sawtimber, u_j is the error term that follows a normal distribution with mean=0 and variance= ϕ^2 . After obtaining OLS estimated coefficients, $\hat{\alpha}_0$, $\hat{\alpha}_1$, and $\hat{\phi}^2$, from equation (1), we can calculate Ornstein-Uhlenbeck parameters with the following equation using the OLS coefficients:

$$(2) \quad \hat{a} = \frac{-\log(1+\hat{\alpha}_1)}{\Delta t_d}, \quad \hat{b} = \frac{-\hat{\alpha}_0}{\hat{\alpha}_1}, \quad \hat{\sigma} = \hat{\phi} \left(\frac{2 \log(1+\hat{\alpha}_1)}{\hat{\alpha}_1(2+\hat{\alpha}_1)\Delta t_d} \right)^{1/2}$$

Where a = mean reversion rate, b = long-term level price, σ = volatility of the Ornstein-Uhlenbeck parameters, and Δt = size of time step. From the solution to equation (2), the binomial tree parameters, U , D , and $\theta_U(i, n)$ are calculated by the following equations (See equation A4 in appendix) :

$$(3) \quad U = e^{\hat{\sigma}\sqrt{\Delta t_m}}, \quad D = e^{-\hat{\sigma}\sqrt{\Delta t_m}}$$

$$\theta_u(i, n) = \begin{cases} 0 & \text{if } \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_m})(\hat{b}-\log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} \leq 0 \\ \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_m})(\hat{b}-\log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} & \text{if } 0 < \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_m})(\hat{b}-\log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} < 1 \\ 1 & \text{if } \frac{1}{2} + \frac{(1-e^{-\hat{a}\Delta t_m})(\hat{b}-\log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} \geq 1 \end{cases}$$

2. Calculating risk neutral probability using capital asset pricing model (CAPM)

The risk neutral probability is the likelihood of future outcome under the assumption that underlying risk asset has the same expected return as riskless assets such as Treasuries bills (Hull 2008). Capital Asset Pricing Model (CAPM) can be applied to calculate the risk neutral probability. The risk neutral probability Π_U is calculated by subtracting a Market Risk Premium adjustment (MRP_{adj}) from the valuation binomial tree's probability θ_U (Guthrie 2009):

$$(4) \quad \Pi_U = \theta_U - MRP_{adj}, \text{ and} \\ \Pi_D = 1 - \Pi_U.$$

The MRP_{adj} is obtained by regressing returns on the market portfolio (Guthrie 2009). The common stock indices such as S&P 500 and NASDAQ are widely used as a proxy for the market portfolio. This study uses the S&P 500 index as a proxy of the market portfolio.

3. Binomial tree of valuation movement

The forest value in each node is denoted by $V(i, n)$, and $V(i, n)$ is related to timber price movements $\theta_U(i, n)$ and $\theta_D(i, n)$. The two-step valuation binomial tree ($n=2$) is shown in Figure A2. The forest value could be increased with probability $\theta_U(i, n)$ or decreased with probability $\theta_D(i, n)$. n is the time step (year) and i is the number of down movements. The risk neutral probability can be expressed as $\Pi_U = \theta_U - MRP_{adj}$ and $\Pi_D = 1 - \Pi_U$. The two-step valuation binomial tree with risk neutral probability is shown in Figure A2. The valuation binomial tree is calculated backwards starting from $V(i, N)$ where N denotes the terminal time step and the ending is $V(0, 0)$. Therefore, valuation at node $V(i, n)$ is

$$(5) \quad V(i, n) = \frac{\Pi_U V(i, n+1)}{R_f} + \frac{\Pi_D V(i+1, n+1)}{R_f}$$

where $R_f = (1 + \text{discount rate})$.

At node (i, n) , the forestland owner faces two alternative situations. The first alternative is harvesting. If she/he decides to harvest the forest, she/he must pay the harvest cost H per timber volume. Total costs are equal to $HQ(n)$ where $Q(n)$ is the total volume of the timber harvested. She/he gains some revenue from selling the timber, which is equal to $X(i, n)Q(n)$, where $X(i, n)$ indicates the expected market timber price in the n th time period. After harvest, the forestland is turning into bare land worth B per hectare. B is the bare land value after harvest. She/he also must pay taxes at a rate of T . All in all, the harvest payoff equation is

$$(6) \quad (1 - T)(X(i, n) - H)Q(n) + B$$

The second alternative is that the forestland owner decides not to harvest, rather postpones the harvest until an appropriate timber price is going to be

reached. In this case, she/he must pay forest maintenance cost per hectare. After one period, the timber price is going to move either up and down. So the corresponding forest value is either $V(i, n+1)$ or $V(i+1, n+1)$. Thus, the expected payoff from postponing harvest is

$$(7) \quad -(1-T)M_T + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f}$$

for all $n = N-1$ where N is the terminal node and MT is the forest maintenance cost. The payoff at the terminal node is

$$(8) \quad (1-T)(X(i, N) - H)Q(N) + B$$

At each node, the decision to harvest or not harvest is re-evaluated. If the present value of the cash flows from harvesting is larger than the current value of the cash flows from not harvesting at the node, the optimal decision is to harvest at this node. On the other hand, if the present value of the cash flows from not harvesting is larger than the current value of the cash flows from harvesting, the optimal decision is not harvesting at this node. Therefore, the valuation at each node $V(i, n)$ is

$$(9) \quad V(i, n) = \max \begin{cases} (1-T)((X(i, n) - H)Q(n)) + B, \\ -(1-T)M_T + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f} \end{cases}$$

The first line in the max function, equation (9), implies the cash flow from harvesting. On the other hand, the second line represents the cash flow from not harvesting. The forest owner makes a decision by comparing the present values of the corresponding expected future cash flows at every node. This problem is solved by calculating $V(i, n)$ backwards, starting from the terminal node where $n = N$ and ending at $V(0, 0)$.

3. Market value of bare land

The backward procedure is conducted recursively over multiple iterations and each iteration represents one harvest/planting rotation. Calculating the market value of bare land follows these steps: (1) The bare land value is zero when calculating value for the first iteration. (2) After finishing the first iteration, $V(0,0)$ (The market value of the forest at date 0) is obtained. (3) The bare land value is estimated by $B = V(0,0) - (1 - T)G$ which implies $V(0,0)$ minus the cost of replanting the forest, where G is regeneration cost and T is tax rate. This first iteration bare land value implies real option value for a single rotation (the value for single rotation forest with flexibility). When calculating the value of the second iteration, the bare land value derived from the first iteration is used as the new initial value instead of 0. This process is repeated until the bare land values converge. This converged bare land value is the real option value with infinite rotation (value of an infinite rotation forest with flexible harvest).

4. Value of flexibility

The value of flexibility is calculated by comparing bare land value from fixed harvest with the value of real option. The valuation method for fixed harvest follows the same process with real option but assumes that the harvest date is fixed. Suppose that the harvest decision is fixed at node M (e.g., 30 years or any years smaller than the terminal node N (100 year), $M < N$), the terminal condition is $(1 - T)(X(i, M) - H)Q(M) + B$ and the years larger than M are ignored. The terminal condition is still not different from that used in the real option method except that M instead of N is used. However, at all nodes earlier than M , there is no reevaluation of the decision since the harvest date is fixed. Therefore, the decision to "wait" is only at nodes $n < M$ and the recursive equation at nodes $n = (M - 1)$ to $n = 0$ becomes

$$(10) \quad V(i, n) = -(1 - T)M_T + \frac{\Pi_u(i, n)V(i, n + 1) + \Pi_d(i, n)V(i + 1, n + 1)}{R_f}$$

The value of bare land converges to the value under the infinite rotation after certain number of iterations. This value is the Land Expectation Value (LEV) of in-

finite rotation (Tee et al. 2014). The difference between LEV and real option (flexible harvest decision) value is the value of flexibility.

III. Application of real option to flexible harvest decision

Forests play a significant role in carbon sequestration because trees absorb carbon during growth. Several studies (Alavalapati and Kant 2014; Tee et al. 2014; Petrusek and Perez-Garcia 2010) have asserted that we should consider forests not only as a source of timber but also a carbon pool. Therefore, the stock of stored carbon in trees should be considered when we choose the optimal harvest age. Many studies have examined the relationship between optimal harvest age and carbon storage ability to stand trees, but most analyses have focused on carbon sequestration only in living trees. Dead trees, however, represent a significant proportion of the total carbon stored in a forest (Asante and Armstrong 2012). Therefore, stored carbon by dead trees may be necessary when determining optimal harvest age. This study aims to establish three different real options models to compare optimal harvest ages and bare land prices.

1. Timber only

The valuation function for timber only is the same as equation (11) discussed in the previous section:

$$(11) \quad V(i, n) = \max \left\{ \begin{array}{l} (1-T)((X(i, n) - H)Q(n)) + B, \\ -(1-T)M_T + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f} \end{array} \right.$$

The terminal node N is 100 years and the results for the rotation ages of up to 90 years will be reported.

2. Timber and carbon storage in living trees

Carbon of trees provides additional benefit to forest owners. Carbon benefits are usually considered the amount of carbon per unit volume of biomass (Amacher, Ollikainen, and Koskela 2009). Since $Q(n)$ as a growth function of a forest at time n and Q_c as the carbon stock (t/ha) in the forest of volume $Q(n)$, the change in the benefit from sequestered carbon in living trees is a function of time n : $X_s([Q_c(n) - Q_c(n-1)])$ where X_s is the social cost of carbon. The stored carbon in standing living trees is derived from a forest ecosystem yield table. The forest ecosystem yield table (J. Smith et al. 2006) provides tabulated carbon density at different stand ages and timber volumes by carbon pools including live trees, standing dead trees, soil organic matters and so on. If timber age or volume is not explicitly provided in the table, the carbon stock is estimated using an interpolation method. The real option valuation function for carbon sequestration by trees is:

$$(12) \quad V(i, n) = \max \left\{ \begin{array}{l} (1-T)\{(X(i, n) - H)Q(n) - X_s Q_c(n-1)\} + B, \\ - (1-T)(-M_T + X_s[Q_c(n) - Q_c(n-1)]) \\ + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f} \end{array} \right\}$$

3. Timber and carbon storage in living trees and dead trees damaged by SPB

The SPB infestation risk affects both the amount of carbon sequestration in trees and timber/wood products per unit forest land area. The trees killed by SPB have a lower merchantable value and preserve less carbon than healthy trees, but these dead trees still represent a substantial proportion of the total carbon stored in forest stands (Asante and Armstrong 2012) and can/will be replaced by new trees naturally and with human assistance. Assume that the percent of trees killed by SPB in each year is given by $\delta\%$. The forest owners may clear cut or damaged trees in the same year or delay the harvest to a future year. In this case, one

should separate the two carbon sequestration pools: 1) carbon pool from live standing trees, and 2) carbon pool from trees killed by SPB.

The timber production in year n will decrease due to SPB damage. Assume that the average yearly SPB damage is given by $\delta\%$, then the total timber production (m^3/ha) in year n will decrease according to equation (13). Therefore, the total tree production will be $Q^*(n)$ instead of $Q(n)$ as given below:

$$(13) \quad Q^*(n) = Q(n) - \delta Q(n).$$

The value of the live standing tree pool is

$$(14) \quad X_s [Q_c^*(n) - Q_c^*(n-1)].$$

Equation (14) implies the value of carbon stored in live standing trees in each year. $[Q_c^*(n) - Q_c^*(n-1)]$ is carbon density (t/ha) and X_s is the social cost of carbon (\$/t). Assume that average yearly SPB damage is given by $\delta\%$, then the total volume of live trees on the site in year n is $Q^*(n) = Q(n) - \delta Q(n)$. The carbon density stored in live trees, $Q_c^*(n)$ is calculated from the forest ecosystem yield table with the corresponding volume $Q^*(n)$ using an interpolation method.

The damaged tree pool (DTP) implies carbon stored in standing dead trees killed by SPB. The trees killed by SPB are assumed to decompose at a rate of η per year, and trees killed by SPB are added to the DTP each year. Therefore, the DTP pool grows according to

$$(15) \quad D(n+1) = (1-\eta)D(n) + \delta Q(n+1)$$

where $D(n)$ represents carbon stored in the damaged tree pool. The estimated decomposition rate is $\eta=0.00578$, which is derived from Asante, Armstrong, and Adamowicz (2011). δ is the average SPB risk. The change in DTP for the no harvest case is $\Delta D(n) = -\eta D(n) + \delta Q(n)$, which implies

$\Delta D(n) = -\eta e^{rn} (D(0) + \sum_{0}^n e^{-r\eta} Q(n-1)) + \delta Q(n)$ where r = the discount factor. Combining all the equations stated above yields the real options value function under SPB risk:

$$(16) \quad V(i, n) = \max \left\{ \begin{array}{l} (1-T)\{(X(i, n) - H)Q^*(n) - X_s[Q_c^*(n-1) + D(n-1)]\} + B, \\ -(1-T)(-M_T + X_s([Q_c^*(n) - Q_c^*(n-1)] + \Delta D(n))) \\ + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f} \end{array} \right\}$$

Starkey et al. (1997) examined that SPB infected at least 10 percent of the slash and/or loblolly pine forest in the southern U.S. Reed (1979) simulated the spread of SPB infestation using a nonlinear spot growth model. He tested the model on 11 infestation spots from northern Georgia and projected 6% of the total number of tree killed by SPB. However, it was not very precise model to estimate damages from individual infestation (Thatcher 1981). There are not many studies to investigate the SPB infestation in loblolly pine forest only and previous studies cannot reflect the current trend of SPB infestation in loblolly pine forest. With this limitation, this study assumes 3% of SPB damages. This number may reflect the current overall trend of SPB infestation risk in the southern U.S. Because the SPB risk δ is assumed to be constant, sensitive analysis will be performed.

IV. Data and cash flows

1. Timber volume and mean carbon stock in the South and South Central region

The mean volume of timber growth and estimated carbon stock for loblolly pine in the southern U.S. are shown in Figure A4 and Figure A5 in the appendix, respectively. The mean volume of timber growth and the estimated forest carbon stock of southern (or loblolly) pines are obtained from "Forest Ecosystem Carbon Tables" from the USDA Forest Service (J. Smith et al. 2006). The Tables were developed using a national-level forest carbon accounting model (FORCARB2), a timber projection model (ATLAS), and the USDA Forest Service and the Forest Inventory and Analysis (FIA) Program's database on forest survey (FIADB) (J. Smith et al. 2006).

2. Costs and cash flows

Forest management costs and cost cash flows are shown in Tables A1 and A2 in the appendix. These costs are based on market research (Doran et al. 2009).

Carbon stocks are calculated based on the timber volume for the loblolly pine forest (living and dead trees, m^3/ha) using the forest carbon table in "Methods for Calculating Forest Ecosystem and Harvested Carbon with Standard Estimates for Forest Types of the United States" (J. Smith et al. 2006). The average stumpage price of southern pine sawtimber price movement is shown in Figure A6 in the appendix and \$150 is the long - term level price of southern pine sawtimber stumpage price calculated by equation (2). The timber stumpage price is an ideal state variable for calculating forest value because the timber stumpage price is the price of timber while it is still standing. So the stumpage price does not reflect the additional cost such as cost of harvesting and transporting the log to the mill (Guthrie 2009). The social costs of carbon (Figure A7 in the appendix) used in the model are obtained from the Interagency Working Group's Technical Support Report (Council of Economic Advisers et al. 2013).

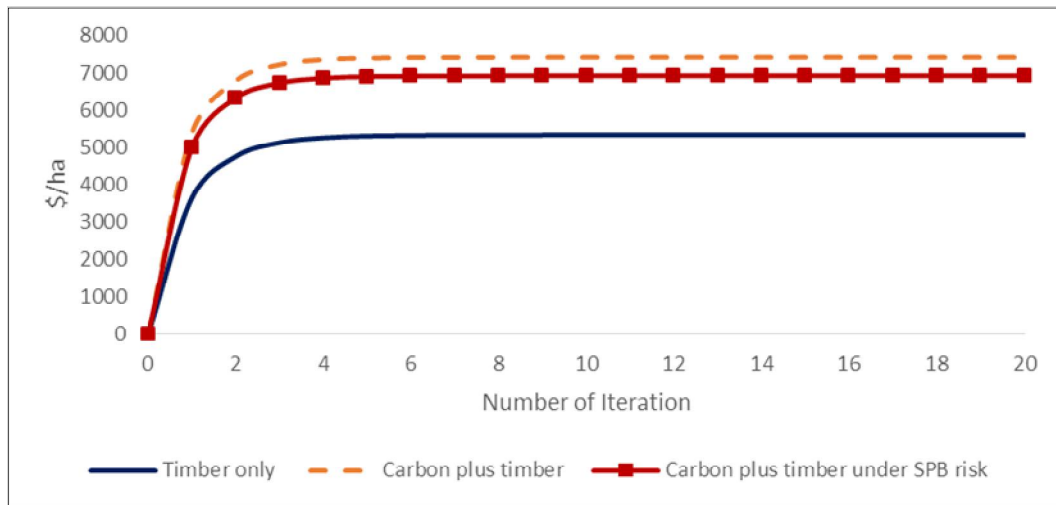
V. Results

1. Land value (real option), harvest threshold and value of flexibility

The results for the flexible harvest (real option) of infinite rotation are shown in Figure 1. For the timber only cases, the bare land value converges to \$5329/ha, after nine cycles/rotations of harvest-and-replant. For the timber plus carbon case (\$75/ha of carbon cost is assumed), the bare land value converges to \$7408/ha, after eight cycles of harvest-and-replant. For the case considering damage of SPB case (a 3% of SPB damaged is assumed), the bare land value converges to \$6918/ha, also after eight cycles of harvest-and-replant. To consider the carbon storage ability of forest, the forest value would increase by 39%, compared to the case of considering only timber price. The SPB risk would decrease the forest value. The bare land value damaged by SPB would decrease by 6% compared to the case of the timber plus carbon forest. However, the SPB damaged forest still

has a higher value than the timber only case because even if SPB damages the forest, the forest still has the ability of carbon storage. Thus, the value of carbon storage would compensate the price loss from damaged timber by SPB.

Figure 1. Infinite rotation values for bare land



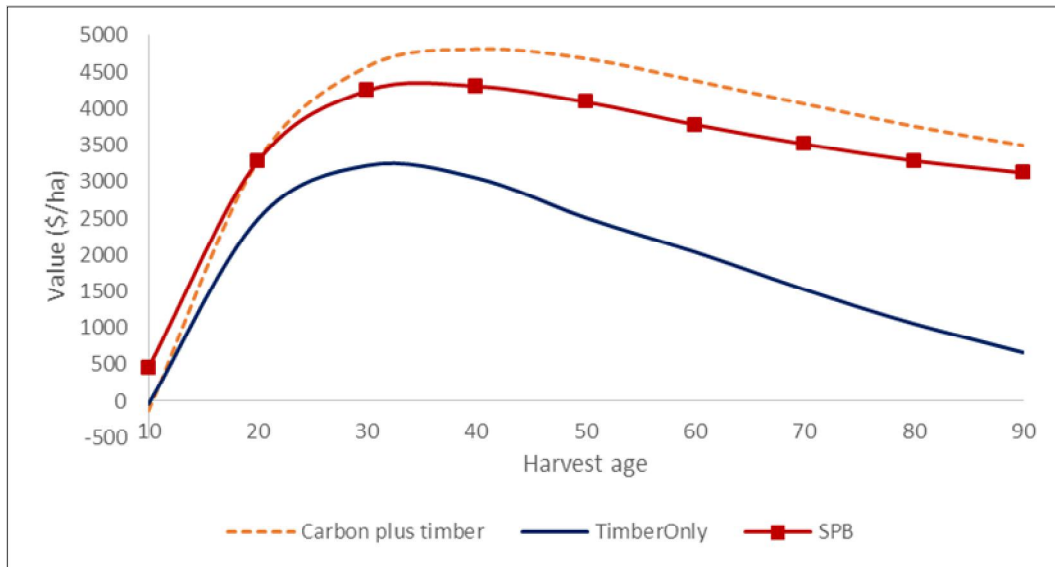
The market value of forests for fixed harvest of infinite rotation is given in Figure 2. The infinite rotation problem is commonly known as the Faustmann rotation, which is defined as "choosing the harvest period to maximize the net present value of a series of future harvest" (Grafton et al. 2008, 138; Gane, Gehren, and Faustmann 1968). In this study, the NPV of a forest could be indicated as a sum of discount net cash flow over an infinite time horizon (Viitala 2006). For evaluating the value of forests for fixed harvest, the same process is used with flexible harvest, but the fixed harvest case assumes that the harvest decision is fixed at the node $t = \text{fixed harvest age}$. Thus, the backward evaluations are started from node t (e.g.: 60 years, 50 years) rather than N (100 year), without no re-evaluation of a harvest decision. Thus, the valuation equation for each node equals to equation (17) and value of bare land converges to infinite rotation NPV of the fixed harvest.

$$(17) \quad V(i, n) = -(1-T)M_T + \frac{\Pi_u(i, n)V(i, n+1) + \Pi_d(i, n)V(i+1, n+1)}{R_f}$$

Under the fixed harvest assumption, generated net present value (NPV) of the forest by timber only case is, \$3220/ha, around age 30. In timber plus carbon case, the net present value of forest is at its maximum, \$4812/ha, at age 40. In the case of timber plus carbon under SPB risk, the net present value of the forest is the highest, \$4308/ha, at age 40. If allowed for flexible harvest (real option), the market value of the bare land is \$5329/ha for the timber only case, \$7408/ha for the timber plus carbon case, and \$6918/ha for the case of timber plus carbon under SPB risk, respectively. Thus, timber harvest flexibility adds approximately 65% to the value of bare land for the timber only case (54% for the timber plus carbon case, 61% for the case of timber plus carbon under SPB risk). This result shows that flexible harvest generates the higher valuation through allowing forest owners to make a better investment decision using information of various price levels. If timber prices are low, the forest owners can postpone harvest while they hasten harvest when prices are high.

Using these results, we can estimate the optimal harvest/rotation age as well. The NPV of the forest is maximized at the point of optimal rotation age for both fixed rotation and infinite rotation. The optimal rotation age is 30 years for the timber only case, 40 years for the timber plus carbon case and 40 years for the case of timber plus carbon forest under SPB risk. The optimal rotation age increases when considering the carbon storage ability of the forest. In the case of SPB damage, the optimal rotation is similar to the carbon forest case, but the forest value is lower than that under the carbon forest case at the optimal rotation age. The value of flexibility also increased if we consider carbon storage ability of the forest because the capacity to be flexible can increase the value of investment when uncertainty and irreversibility become larger (Tee et al. 2014).

Figure 2. Market value of bare land: fixed harvest, infinite rotation



Figures 3-5 show the optimal harvest threshold for infinite rotations, timber only case, carbon plus timber case and carbon plus timber under SPB risk. The values are rounded off to the nearest whole number. These figures show the harvest threshold price for all possible ages of the forest. The shaded area implies the range of sawtimber price that is optimal to harvest for a given forest age. In every case, if the forest is very young (less than 10 years old), the optimal choice is not to harvest unless the timber price would become extremely high. However, as the age of the forest increases, the threshold price falls. For example, in Figure 3, if the timber price is above $\$258/m^3$ when the forest age is between 20 to 26 years old, the optimal decision is to harvest while the optimal decision would be to defer harvest if the timber price is below $\$258/m^3$.

Figure 3. Sawtimber threshold prices for the timber-only case

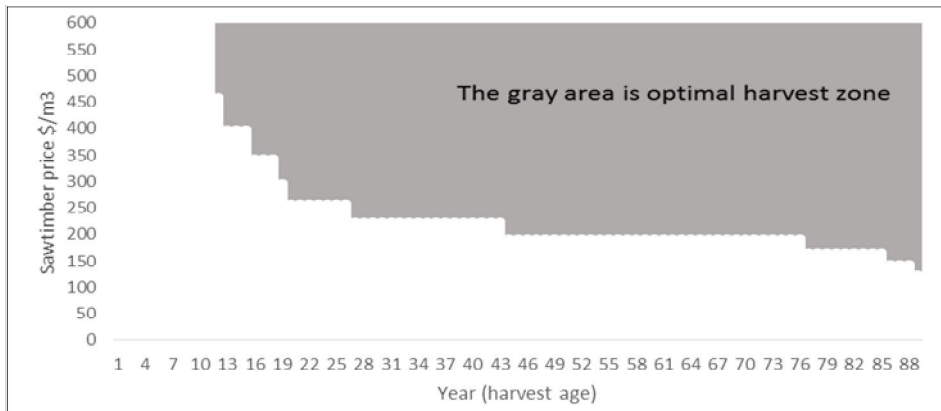


Figure 4. Sawtimber threshold prices for the carbon-forest case

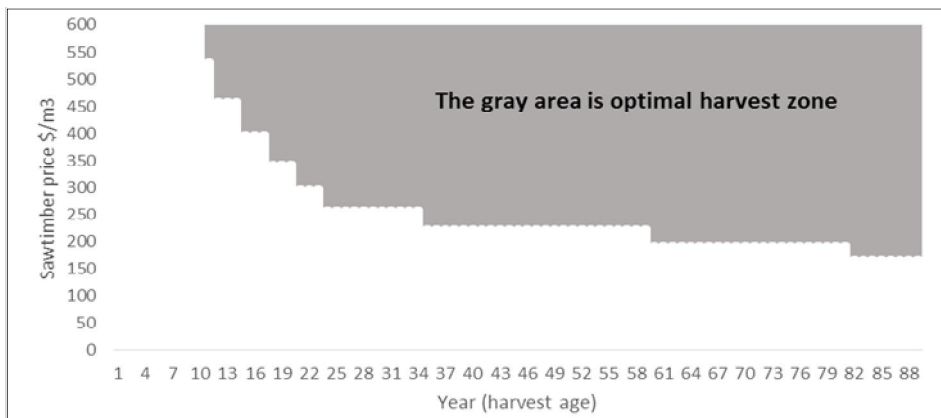


Figure 5. Sawtimber threshold prices for the carbon-forest under SPB risk case

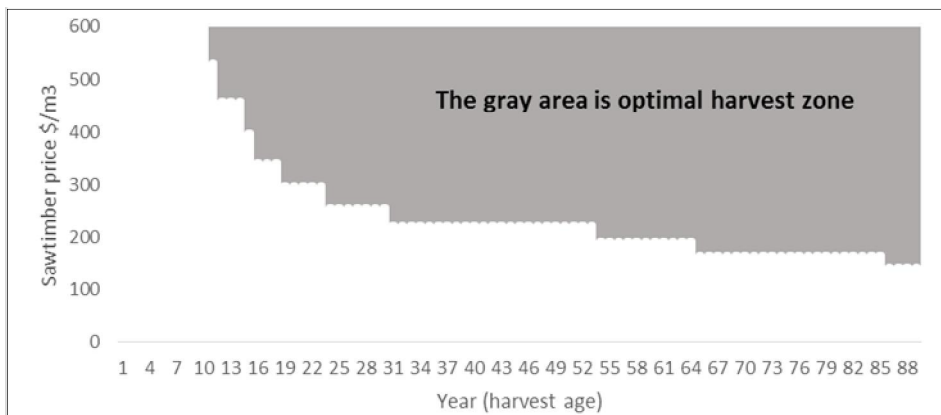


Figure 6 and Table 1 compares the timber threshold price changes among timber only case, carbon plus timber case, and carbon plus timber under SPB case for all possible ages of the forest. It is apparent from this figure and table that harvest threshold price decrease as trees age for all three cases. If the age of trees is younger than 10 years, the optimal decision is not to harvest in all cases. The threshold price tends to be high under the cases with considering carbon storage, compare to timber only. This is because carbon store ability of forest incurs a higher opportunity cost of harvesting the forest, therefore, to offset the burden of harvest, a higher timber price (revenue) would be required compare to timber only case. The SPB damage reduces the advantage of standing forest its threshold price is higher than timber only case because dead trees still provide carbon sequestration. The benefit from carbon sequestration of standing tree partially compensates the lost from reducing the total volume of harvest by SPB damage.

Figure 6. Comparisons of threshold price changes: timber only vs. carbon forest under SPB vs. carbon forest

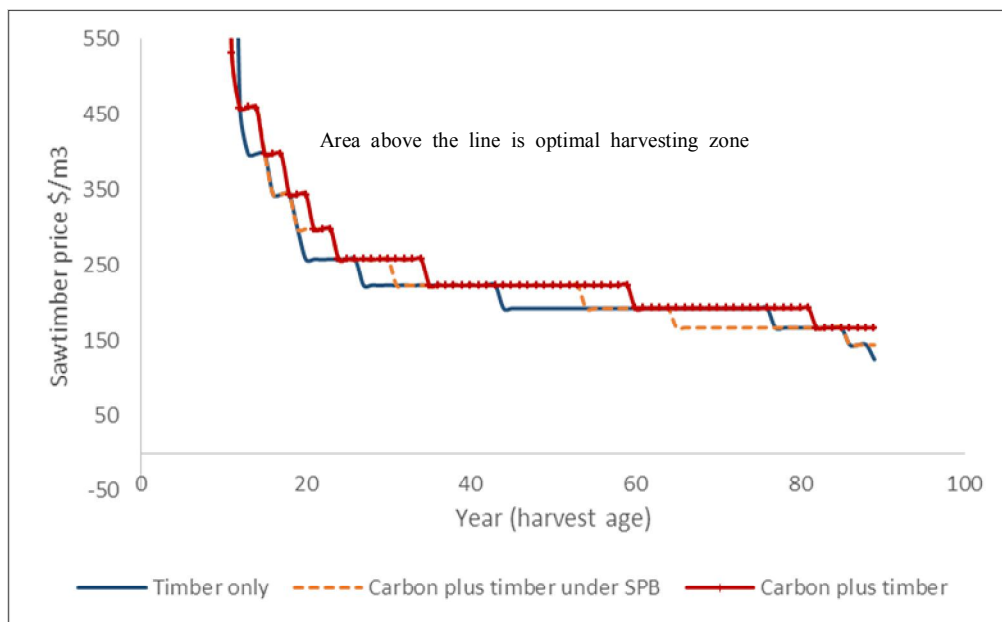


Table 1. Comparison of timber threshold price ages

| Age | Timber only | Carbon Forest | Carbon forest under SPB risk |
|-----|-------------|---------------|------------------------------|
| 10 | Not harvest | 1000 | 1000 |
| 11 | 460 | 531 | 531 |
| 20 | 258 | 344 | 297 |
| 40 | 223 | 258 | 227 |
| 60 | 193 | 223 | 193 |
| 80 | 167 | 223 | 167 |
| 86 | 144 | 167 | 144 |
| 89 | 125 | 167 | 144 |

2. Sensitive analysis for carbon social cost

Figure 7 presents infinite rotation valuation for fixed harvest under various levels of social cost of carbon. As the social cost of carbon increases from \$50/t to \$75/t, the expected NPV of the forest increases from \$4224/ha to \$5164/ha at 2.4% discount rate. The optimal rotation age does not change, but the bare land values changes according to difference of carbon social cost; as the social cost of carbon increases, the value of the forest increases.

Figure 7. Market value of bare land under various levels of social cost of carbon: Fixed Harvest

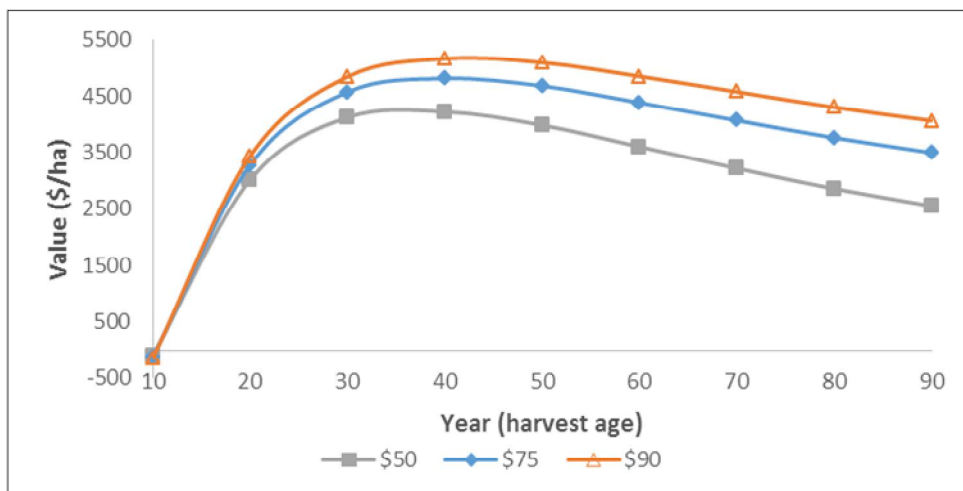
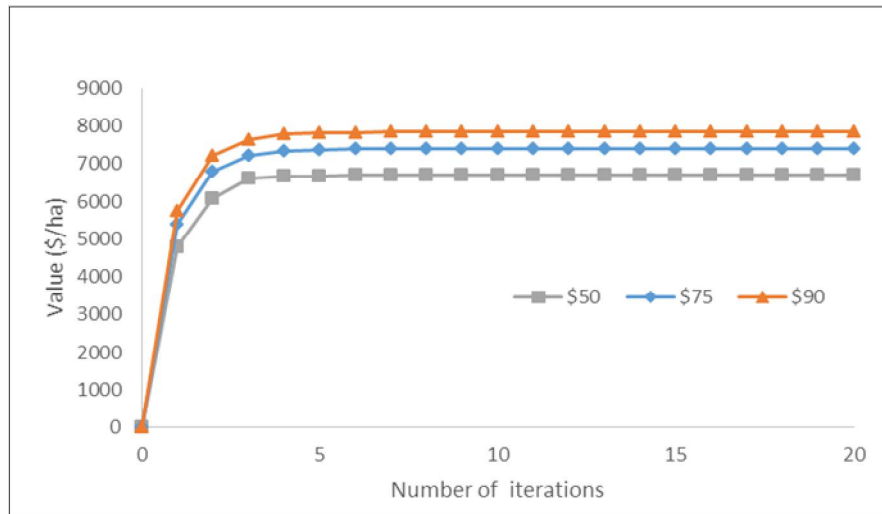


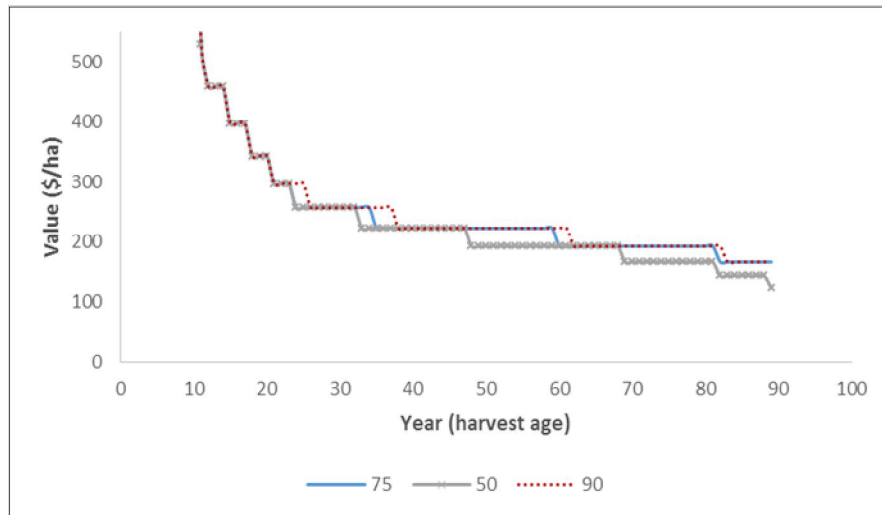
Figure 8. Market value of bare land under various social costs of carbon: Flexible harvest



The bare land price changes for flexible harvest (real option) of infinite rotation under various levels of social cost of carbon are shown in Figure 8. If the carbon social cost is \$50/t, the bare land value converges to \$6699/ha, after eight cycles of harvest-and-replant. If the carbon social cost is \$75/t, the bare land value converges to \$7408/ha, after eight cycle of harvest and replant. If the carbon social cost is \$90/t, the bare land value converges to \$7841/ha, after eight cycle of harvest-and-replant. Compare to fixed harvest case, flexibility adds approximately 59% to the value of bare land under a \$50/t social cost, 54% under a \$75/t social cost, 51% under a \$90/t social cost.

The timber threshold price changes for all possible ages of the forest under various level of social cost of carbon are presented in Figure 9. The harvest threshold price decreases as the social cost of carbon decreases. No difference in threshold price depending on social cost of carbon if the age of the forest is young (less than 20 years old). If the forest age is 36 years, the timber threshold price is \$257/m³ for a \$90/t of carbon social cost, \$223/m³ for a \$50/t of carbon social cost, and \$223/m³ for a \$75/t of carbon cost, respectively. The timber threshold price decreases as the trees grow. The higher social cost of carbon increases the opportunity cost to harvest trees. Therefore, it requires a higher timber price is necessary to compensate the loss of the opportunity cost associated with cutting trees down. Therefore, as the carbon social cost increases, the forest owner would consider delay timber harvest if anything else remains the same.

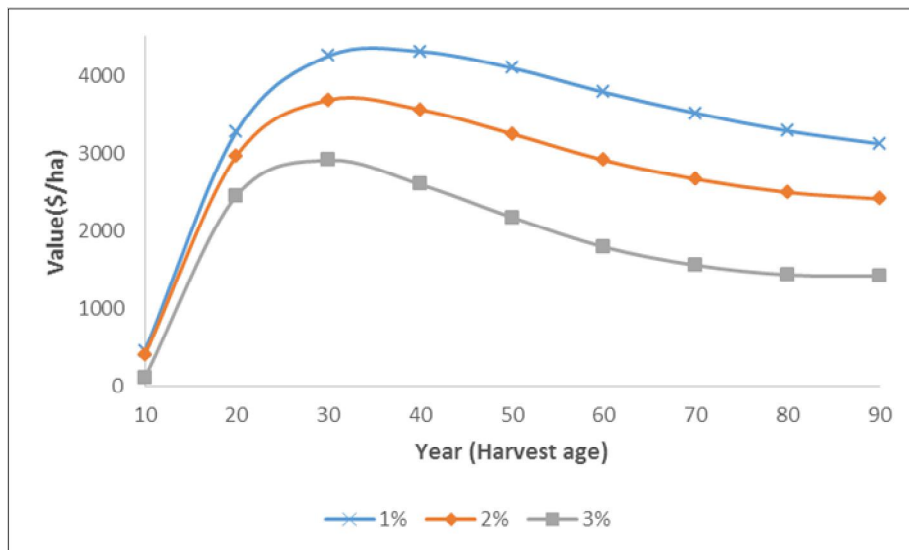
Figure 9. Timber threshold price by different social costs of carbon



3. Sensitivity analysis for SPB risk

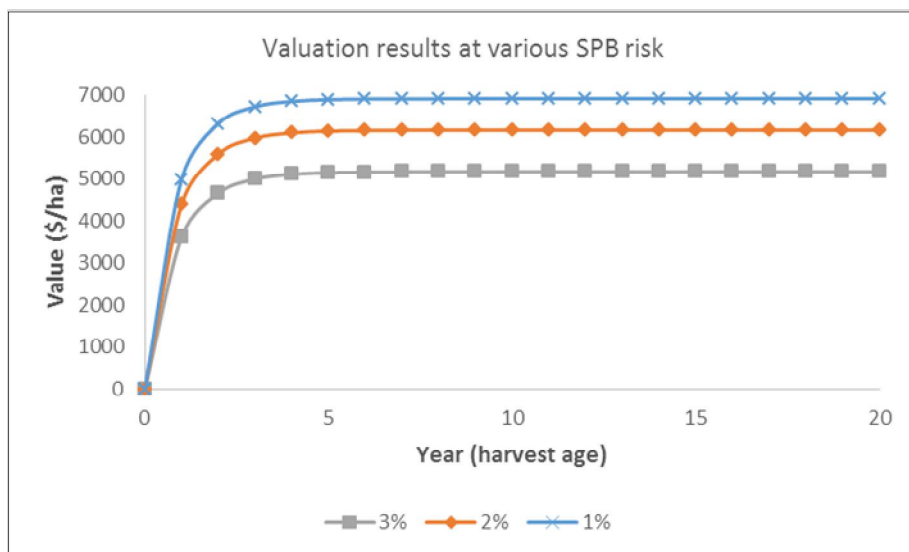
Fixed harvest valuation (infinite rotation) under various SPB damage rates are illustrated in Figure 10. If the SPB damage rate increases, the value of bare land will decrease. If SPB damages 1% of the forest, the forest value is \$4308/ha at the optimal rotation age (40 years old). However, if SPB damages 2% of the forest, the forest value is \$3681/ha at the optimal harvest age (30 years old). If SPB damages 3% of the forest, the forest value is \$2908/ha at the optimal harvest age (30 years old). As the SPB risk increases, both the bare land value and the optimal rotation age decrease because high SPB infestation reduces both total harvest volume and carbon sequestration ability of trees. This generates a potential profit loss to the forest owners by reducing timber productivity in forest. When forest owners make a harvest decision, they need to determine if the rate of return from continuing the investment in the forest is worth more than the rate of return received from an alternative investment (Jacobson 2015). Therefore, incentives from continuing to grow the trees would decrease under high SPB infestation risk by decreasing the future expected rate of return from continuing the investment in the trees. Thus, forest owner's choice is seeking other opportunities to invest.

Figure 10. Value of bare land (with fixed harvest)at various SPB risks



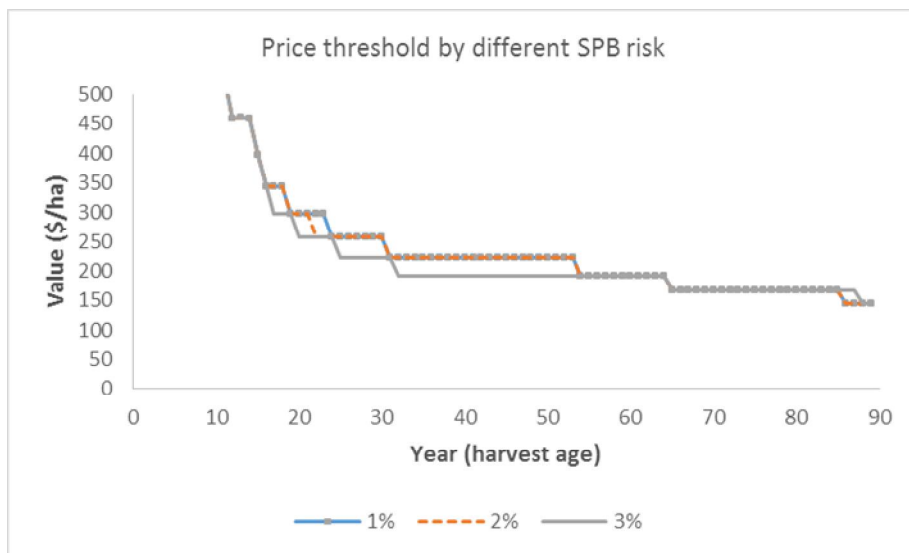
The change of real option value (flexible harvest valuation) under various SPB damage rates are shown in Figure 11. The real option values decrease from \$6918/ha to \$5169/ha as SPB risk rises from 1% to 3%.

Figure 11. Market value of bare land (flexible harvest) changes at various SPB risks



The timber threshold price for harvesting at various SPB damage rates. For example, the harvest threshold price is \$297/m³ at a 1% SPB risk, \$258/m³ at both 2% and 3% SPB risk at age 23 is shown in Figure 12. If the forest age is 40 years, the threshold price is \$223/m³ at 1% and 2% SPB risk, \$193/m³ for the case of 3% SPB risk. A higher SPB risk reduces the benefit from keeping the forest. Therefore, harvesting is optimal at a lower timber price as SPB damage risk becomes more severe, especially, if the forest is younger than 55 years.

Figure 12. Optimal harvest price flow at various SPB damage rates



VI. Conclusion

This paper evaluates the combined impact of the three factors on forest manager's decision making using real option approach. The major finding of this paper is that flexible harvest decision making using real options is a better strategy than the fixed harvest decision when forest owners face several uncertainties including sawtimber price volatility, climate change, and insect outbreaks. A higher bare land value is generated if a flexible harvest decision making (real option) by incorporating stochastic price movement is allowed because the value of flexibility

adds to forest values when flexible harvest decision is allowed. The CO₂ storage of a forest enhances the bare land value while SPB outbreaks reduce the bare land value. However, if we consider the carbon sequestration ability of damaged trees, the bare land value is still higher than that without taking into account carbon storage of damaged trees. The value of standing trees is higher as the carbon social cost increases due to increasing opportunity cost of carbon sequestration on trees. When social cost of carbon is high, the incentive from converting abandoned agricultural land to forest land and using wood products instead of other material will become higher. Moreover, the high social cost of carbon also adds value to wood products because the wood products also contribute to carbon storage.

As the global CO₂ concentration increases under climate change, the value of carbon storage of forest would increase. Therefore, at higher social cost of carbon, higher timber price is required to warrant harvesting due to increasing opportunity cost of cutting trees. Higher SPB risk tends to reduce the bare land value of forest. The high bare land value of carbon forest provides an incentive to forest owners to plant new forests and perform intensive treatments to keep forests healthy and productive. The U.S. forests currently absorb 10% of the national greenhouse gas emissions (Ingerson 2009). Increasing the forest rotation age by increasing the value of standing trees could enhance forests' CO₂ storage by deferring harvest. This might provide positive impacts on CO₂ mitigation in the southern U.S. This study confirms that standing forests could provide social benefits by absorbing CO₂. However, planting new forests and keeping them healthy may require additional costs such as the cost of pesticide and fertilization. This might carry an extra burden to forest owners. Therefore, policy makers should establish legislation that provides additional incentives to forest owners to offset additional burden by differing harvest and planting new forest. Emissions trading may be one of the solutions. Under emissions trading, the forest owners could earn carbon credit by standing forest and sell them in domestic and international market. For example, under the the New Zealand Emission Trading Scheme (NZETS), the post-1989 forests (planted on and after 1st January 1990) are qualified as carbon credit that could be accumulated or immediately sold in carbon market (Tee et al. 2014). This could provide extra income to forest owners, and the extra cash flow might generate incentives to forest owner to harvest new forests.

A limitation of this study is the absence of considering various forest management practices including pruning, thinning and fertilizing. Also, the pesti-

cide control impact should be considered in the case of SPB outbreak risk in future research. The impact of CO₂ fertilizations on forest productivity might be included in real option valuation equations as well. The increments of timber products because of CO₂ fertilizations may offset the loss from timber damages by SPB infestation under climate change. To consider these factors, more sophisticated real option valuation modeling approaches will be necessary for further studies. Unless several limitations, I convince that the paper will give insights into what forest owners need to do for improving their timber harvest decisions under uncertainty. The optimal harvest thresholds in particular, provide a useful guideline for forest owners by offering an insightful decision-making tool which can be compared with actual timber price in every year.

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APPENDIX

Figure A1. Two-step price binomial tree

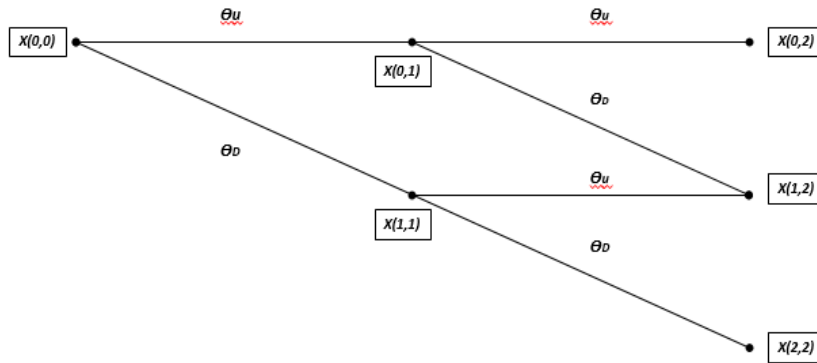


Figure A2. Two step valuation binomial tree

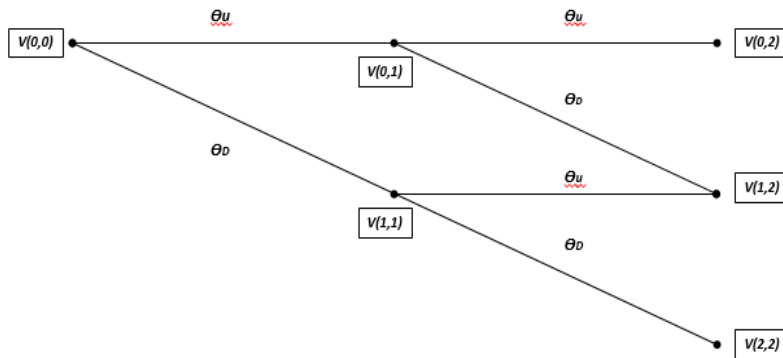


Figure A3. Two step valuation binomial tree with risk neutral probability

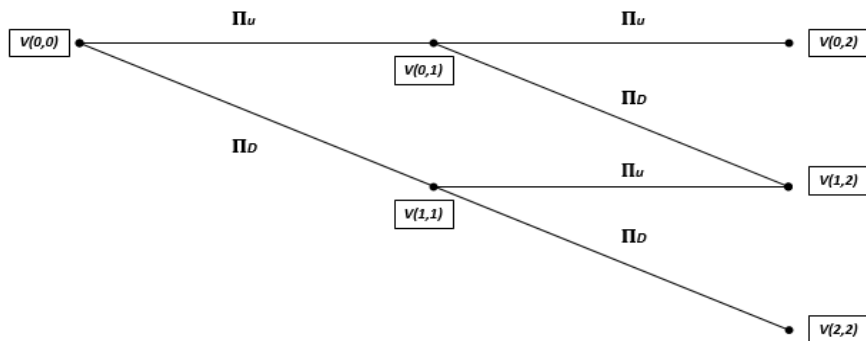


Figure A4. Estimates of timber volume for loblolly pine stands in southern U.S.

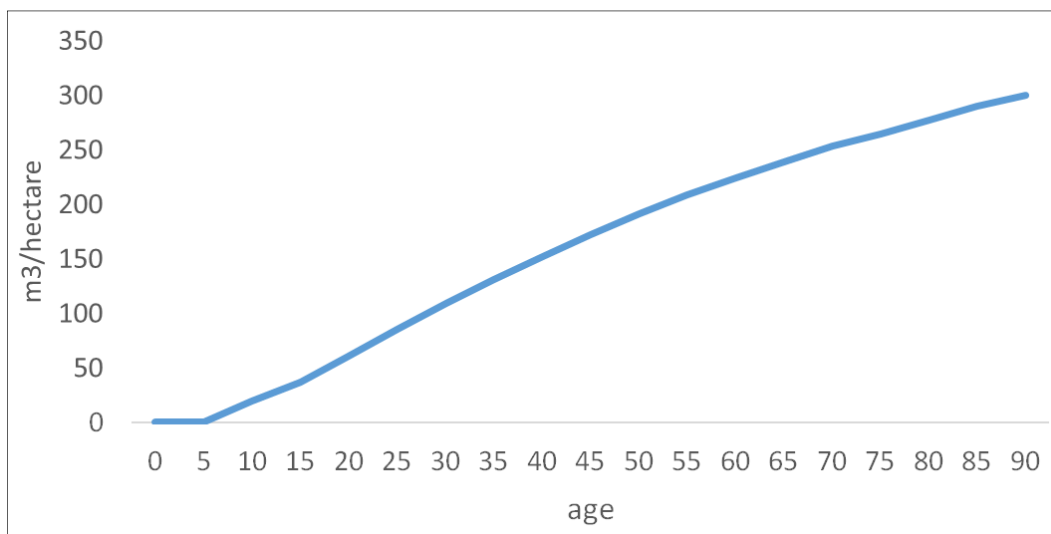


Figure A5. Estimates of carbon stock for loblolly pine stands in southern U.S.

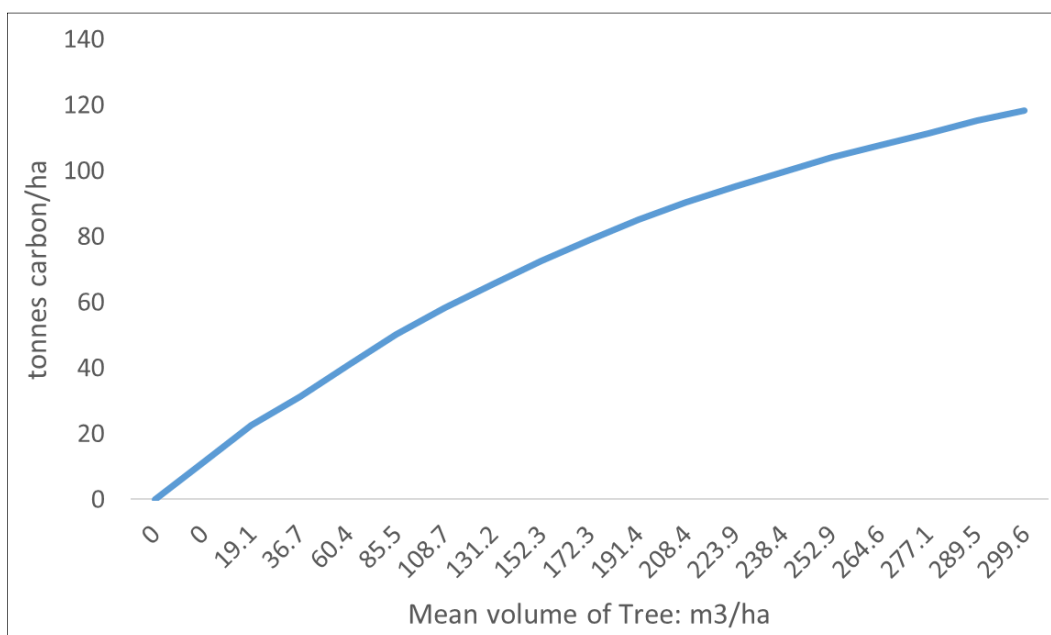


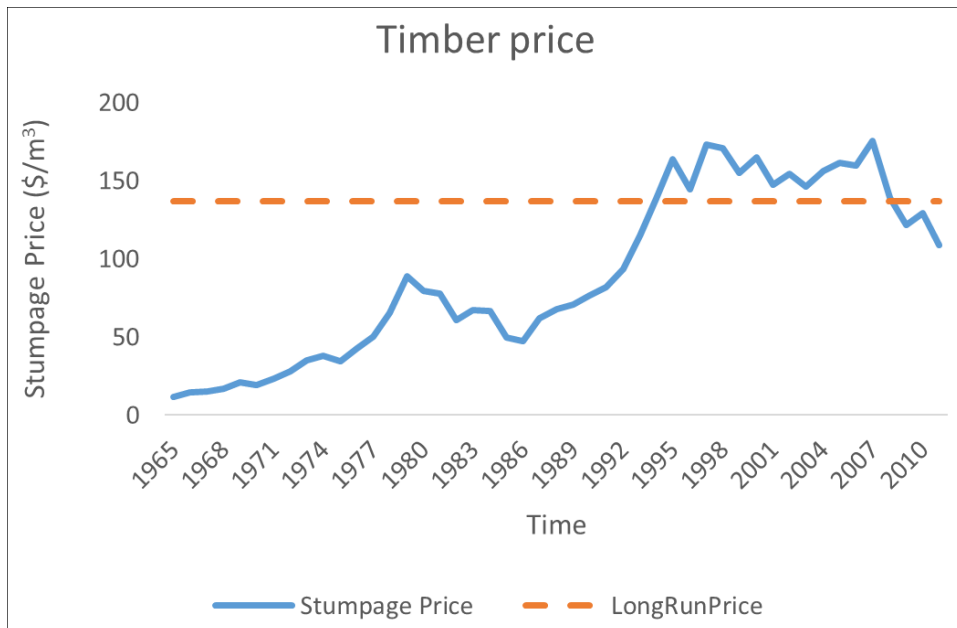
Table A1. Forest management costs

| Management cost description | Cost (\$) |
|--|------------------------|
| Regeneration cost (including the cost of site preparation, seedling, planting and weed control), G | \$618/ha |
| Forest management cost, M_T | \$22/ha |
| Tax rate, T | 28% |
| Harvest cost H_T | \$68.67/m ³ |
| Discount factor (Risk free interest ate base on current 20 year U.S. treasury rate), r | 2.5% |

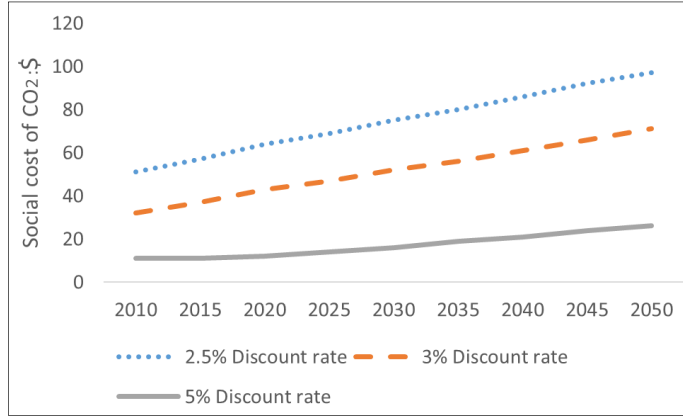
Table A2. Cost cash flow

| Year | 0 | 1 | ... | 15th rotation | ... | 24th rotation | ... | 90th rotation |
|-------------------------|-------|-------|-----|------------------|-----|------------------|-----|------------------|
| Planting Cost | (618) | (618) | ... | (618) | ... | (618) | ... | (618) |
| Maintenance Cost, M_T | (22) | (22) | ... | (22) | ... | (22) | ... | (22) |
| Timber Revenue | \$ | \$ | ... | \$ | ... | \$ | ... | \$ |
| Harvest Cost | \$ | \$ | ... | \$ | ... | \$ | ... | \$ |

Figure A6. Average stumpage price of sawtimber



(Source: Louisiana Department of Agriculture)

Figure A7. Revised social cost of CO₂, 2010 – 2050 (in 2007 dollars per metric ton of CO₂)

(Source: Council of Economic Advisors 2013)

Calculating size of up and down movement U, D

The size of an up and down movement U, D can be obtained from following process (Guthrie 2009). The log price of $X(i, n)$ is defined as $\log X(i, n) = x(i, n)$ and which is composed of the following equation

$$(A1) \quad x(i, n) = \log p_0 + (n-i)(\hat{\sigma} \sqrt{\Delta t_m}) + i(-\hat{\sigma} \sqrt{\Delta t_m})$$

where $\log p_0$ = starting value, $(n-i)(\hat{\sigma} \sqrt{\Delta t_m})$ = effect of up moves, $i(-\hat{\sigma} \sqrt{\Delta t_m})$ = effect of down moves. Taking exponentials of both sides of this equation explain that the level of the price at node (i, n) and the up/down moves takes the price to

$$(A2) \quad \begin{aligned} X(i, n) &= e^{x(i, n)} = P_0 e^{(n-2i)\hat{\sigma} \sqrt{\Delta t_m}} \\ X(i, n+1) &= e^{x(i, n)} = P_0 e^{((n+1)-2i)\hat{\sigma} \sqrt{\Delta t_m}} = e^{\hat{\sigma} \sqrt{\Delta t_m}} X(i, n) \\ X(i+1, n+1) &= e^{x(i, n)} = P_0 e^{((n+1)-2(i+1))\hat{\sigma} \sqrt{\Delta t_m}} = e^{-\hat{\sigma} \sqrt{\Delta t_m}} X(i, n) \end{aligned}$$

The size of an up and down moves, U and D , at this node must equal to the following equation. The size of up and down moves are constant through the binomial tree.

$$(A3) \quad \begin{aligned} U &= \frac{X(i, n+1)}{X(i, n)} = e^{\hat{\sigma} \sqrt{\Delta t_m}} \\ D &= \frac{X(i+1, n+1)}{X(i, n)} = e^{-\hat{\sigma} \sqrt{\Delta t_m}} \end{aligned}$$

Calculating probability of up and down movement

The probability of an up movement for mean reverting process was calculated using equation from Guthrie's work (2009). The expected value of the change in the log price over next period is equal to the value that is implied by our normalized parameter estimates.

$$(A4) \quad \theta_u(i, n) = \left\{ \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - x(i, n))}{2\hat{\sigma}\sqrt{\Delta t_m}} \right\}$$

$(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - x(i, n))$, is the expected change in the log price, which is the same as the expected value for the Ornstein-Uhlenbeck process. If the current log price is higher than its long-run level, which is $x(i, n) > \hat{b}$, then the price likely moves to the down, which is $\theta_u(i, n) < \frac{1}{2}$. As the log price grows larger, a down move more likely to happen.

Conversely, if the log price is currently lower than its long-run level then an up move is more likely than down move. If $x(i, n)$ is sufficiently large, then $\theta_u(i, n)$ will have negative value. Similarly, if $x(i, n)$ is sufficiently small, then $\theta_u(i, n)$ will be greater than one. However, since $\theta_u(i, n)$ is a probability, the value of $\theta_u(i, n)$ must be located between 0 and 1. Thus, our solution set $\theta_u(i, n)$ equal to 0 if expression in equation (A4) has negative value, and 1 if greater than 1. Therefore, the final form of the probability of an up and down movement at node (i, n) should be

$$(A5) \quad \theta_u(i, n) = \begin{cases} 0 & \text{if } \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - \log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} \leq 0 \\ \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - \log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} & \text{if } 0 < \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - \log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} < 1 \\ 1 & \text{if } \frac{1}{2} + \frac{(1 - e^{-\hat{a}\Delta t_m})(\hat{b} - \log(X(i, n)))}{2\hat{\sigma}\sqrt{\Delta t_m}} \geq 1 \end{cases}$$

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